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Analytical solution of Boudreau's equation for a tracer subject to food-feedback bioturbation

Abstract—Boudreau (1998) obtains a differential equation for a tracer at steady state, subject to mixing associated with bioturbation in a mixed layer of thickness L. As he suggests, the equation may be solved numerically, but an expression for the solution in terms of conventional functional forms would be convenient for the researcher. Analytical solutions corresponding to two different standard boundary conditions are derived below, and their behaviors are discussed briefly.

The steady-state mass balance equation for a tracer subject to the effects of biodiffusion, advective transport, and first-order decay in the mixed layer can be written as follows (Boudreau 1998):

$$\frac{d}{dx}\left(D_B(x)\frac{dC}{dx}\right) - w\frac{dC}{dx} - \lambda C = 0 \tag{1}$$

in which $D_B(x)$ is a mixing or biodiffusion coefficient (length^2 time^{-1}), w = advective transport velocity of the tracer (length time^{-1}), and λ is the first-order decay constant of the tracer (time^{-1}).

A possible form for $D_B(x)$, based on the profile of labile organic matter, which is itself diffusing and fueling bioturbation, leads to the expression (*see* eq. 9 in Boudreau 1998)

$$\frac{d}{dx}\left[D^*G_0\left(1 - \frac{x}{L}\right)^2\frac{dC}{dx}\right] - w\frac{dC}{dx} - \lambda C = 0. \tag{2}$$

Here, D^* is a mixing coefficient per unit concentration labile organic matter (length time^{-1} concentration^{-1}), and G_0 is the labile organic matter concentration at the surface. To obtain the solution to Eq. 2, depth and concentration are transformed as $z = L/(L - x)$ and $C = Y\sqrt{z}e^{Pez/2}$, respectively (cf. Wylie 1966). This results in the equation

$$z^2\frac{d^2Y}{dz^2} + z\frac{dY}{dz} - \left(\frac{1}{4} + Da + \frac{Pe^2z^2}{4}\right)Y = 0 \tag{3}$$

where Pe is Peclet number $= wL/D^*G_o$ and Da is first Damkohler number $= \lambda L^2/(D^*G_o)$.

Boudreau (1986) discusses the significance of these parameters in more detail.

For $Pe > 0$, Eq. 3 is a modified Bessel equation with the following general solution (Abramowitz and Stegun 1964):

$$Y(z) = c_1I_\nu\left(\frac{Pe}{2}z\right) + c_2K_\nu\left(\frac{Pe}{2}z\right) \tag{4}$$

where $\nu = \sqrt{Da + 1/4}$, $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions of order ν of the first and second kinds, and c_1 and c_2 are arbitrary constants determined by the boundary conditions. Thus, the general solution, $C(z)$, can be written

$$C(z) = z^{1/2}e^{Pez/2}\left[c_1I_\nu\left(\frac{Pe}{2}z\right) + c_2K_\nu\left(\frac{Pe}{2}z\right)\right]. \tag{5}$$

Note that as x approaches L, z approaches ∞, and $I_\nu(z) \sim e^z/\sqrt{2\pi z}$, $K_\nu(z) \sim \sqrt{(\pi/2z)}e^{-z}$ (Abramowitz and Stegun 1966), so that $C(z)$ is bounded only if $c_1 = 0$. If the value of C at $x = 0$, C_0, is known, the solution can be written

$$C(x) = C_0\sqrt{\frac{L}{L - x}}e^{Pex/[2(L-x)]}\frac{K_\nu\left(\frac{PeL}{2(L - x)}\right)}{K_\nu\left(\frac{Pe}{2}\right)} \tag{6}$$

and the limiting value of $C(x)$ as $x \to L$ is

$$C(L) = C_0\sqrt{\frac{\pi}{Pe}}\frac{e^{-Pe/2}}{K_\nu\left(\frac{Pe}{2}\right)}. \tag{7}$$

Concentration profiles within the mixed layer for this constant concentration boundary condition for a range of values of Pe and Da are shown in Fig. 1A,B.

A second useful boundary condition for this problem is that of constant flux, f_0, of the tracer at the surface, $x = 0$. The appropriate boundary condition is

$$-DG_0\frac{dC}{dx}\bigg|_{x=0} + wC|_{x=0} = f_0. \tag{8}$$

The solution satisfying Eq. 8 is

$$C(x) = \frac{\frac{2f_0}{w}\sqrt{\frac{L}{L - x}}e^{Pex/[2(L-x)]}K_\nu\left(\frac{PeL}{2(L - x)}\right)}{K_{\nu+1}\left(\frac{Pe}{2}\right) + K_\nu\left(\frac{Pe}{2}\right)\left(1 - \frac{2\nu + 1}{Pe}\right)}. \tag{9}$$

Concentration profiles corresponding to this constant flux boundary condition for a range of values of Pe and Da are shown in Fig. 1C,D. The limiting value of the total flux as $x \to L$ corresponding to the flux boundary condition is

$$f(L) = \frac{2\sqrt{\pi Pe}\,f_0e^{-Pe/2}}{PeK_{\nu+1}\left(\frac{Pe}{2}\right) + (Pe - 2\nu - 1)K_\nu\left(\frac{Pe}{2}\right)}. \tag{10}$$

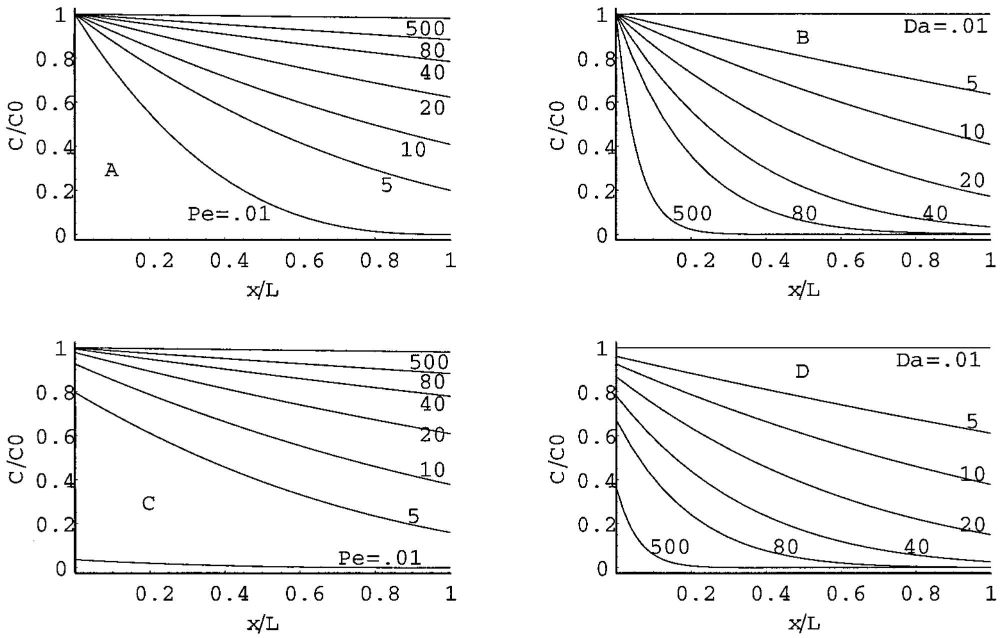

Fig. 1. (A) Tracer concentration profiles in the mixed layer for the constant concentration boundary condition at several values of *Pe,* with $Da = 10$; (B) profiles for the constant concentration boundary condition at several values of *Da,* with $Pe = 10$; (C) profiles as in (A) for the constant flux boundary condition; and (D) profiles as in (B) for the constant flux boundary condition.

Though Eq. 6 and 9 might appear to be unbounded as $x \rightarrow L$ because $L - x$ appears in the denominator of the square root and exponential terms, the limiting behavior of K_ν compensates for this. Numerical implementations of the solutions in the vicinity of $x = L$ should take advantage of this fact (and Eq. 7, 10).

Limiting behavior of the profiles with respect to the parameter values is also of interest. It can be shown that in the limit as $Da \rightarrow 0$ (i.e., conservative tracer) or as $Pe \rightarrow \infty$ (advection dominates biodiffusion), both tracer concentration and flux at any depth in the mixed layer approach their surface values, C_0 and f_0. For $Pe = 0$ (i.e., no advection term), the form of Eq. 3 changes from a Bessel equation to an Euler equation (cf. Boyce and DiPrima 1977), and the concentration profiles for both boundary conditions decrease from the concentrations at $x = 0$ as

$$\left(\frac{L - x}{L}\right)^{\nu-(1/2)}.$$

While a bit cumbersome, the above expressions for *C(x)* can be evaluated with standard spreadsheet programs (at least for integer values of ν), analytical software packages (e.g., Wolfram 1991), or calls to standard mathematical subroutine libraries (e.g., Press et al. 1992). Using the relationships Boudreau (1998) obtained, we find that $L = 9.7$ cm, $Pe = 0.21\ w^{0.4}$, and $Da = 2\ \lambda\ w^{-0.6}$, which permits the calculation of the relative concentration at the mixed depth, L, from the above equations, given only values of λ and w.

Finally, it is important to recognize that Eq. 2 implies that no diffusion occurs at or below depth $x = L$. Assuming that only advective transport occurs below depth L and that the values for λ and w remain constant, then below this depth, both steady-state concentration and flux drop off from their values at depth L, given by Eq. 7 and 10, as $\exp(-\lambda(x - L)/w)$.

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References

ABRAMOWITZ, M., AND I. A. STEGUN. 1964. Handbook of mathematical functions. Dover.

BOUDREAU, B. P. 1986. Mathematics of tracer mixing in sediments: 1. Spatially-dependent, diffusive mixing. Am. J. Sci. **286:** 161–198.

———. 1998. Mean mixed depth of sediments: The wherefore and the why. Limnol. Oceanogr. **43:** 449–457.

BOYCE, W. E., AND R. C. DIPRIMA. 1977. Elementary differential equations and boundary value problems, 3rd ed. Wiley.

PRESS, W. H., S. A. TEUKOLSKY, W. T. VETTERLING, AND B. P. FLANNERY. 1992. Numerical recipes in Fortran: The art of scientific computing. Cambridge.

WOLFRAM, S. 1991. Mathematica. A system for doing mathematics by computer, 2nd ed. Addison.

WYLIE, C. R., JR. 1966. Advanced engineering mathematics. McGraw.

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